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Sequencing games with batch-ordered jobs

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ABSTRACT

The sequencing of jobs on a machine to minimize processing costs is a classical problem in operations research. A key question is how to fairly allocate the resulting total cost among job owners, which is addressed through cooperative game theory. Uncertainty in sequencing often emerges in real-world scenarios when jobs arrive in unordered batches. This paper proposes two new approaches to address such situations and discusses their practical applications. For each approach, we define a cooperative cost game where the worst-case scenario for each coalition is considered. First, we assume that there is no communication among batches, and jobs in a coalition are placed last in their batch. In the second approach, we assume that communication is allowed, and therefore, once jobs are placed last in their batch, they can swap positions if they belong to a connected coalition and the rearrangement leads to cost savings. Finally, we define and characterize rules to distribute the total cost, providing core elements of the corresponding games.

Keywords: Game Theory, Sequencing games, Batch-ordered jobs, Cost allocation.

1. INTRODUCTION

This paper examines cooperative game theoretical approaches to sequencing problems, where a finite set of jobs must be processed on one or multiple machines and the challenge lies in distributing the total processing cost among job owners. The study of sequencing situations in cooperative game theory originates with Curiel et al. (1989), who introduced sequencing games with an initial order of jobs, showing how coalitions could achieve savings through admissible rearrangements and introducing and characterizing the equal gain splitting rule. Subsequent contributions have enriched this framework. For instance, Curiel et al. (1993) analyze additive and weakly increasing cost functions; Hamers et al. (1996) propose the split core; and Slikker (2023) introduce the stable gain splitting rule.

Beyond the initial order model, several extensions have been explored. Hamers et al. (1995) study equal processing times with proportional ready times, Borm et al. (2002) incorporate due-date criteria, Çiftçi et al. (2013) address batch-processing machines; while other works focus on contexts such as family setup times (Grundel et al., 2013) or endogenously chosen numbers of machines (Atay and Trudeau, 2024). More recently, Saavedra-Nieves et al. (2025) investigate position-dependent costs.

In contrast, sequencing situations without an initial order has been studied by Chun (2006) and Klijn and Sánchez Rodríguez (2006), who propose the proportional and cost splitting rules for equal processing times. Gerichhausen and Hamers (2009) introduce partitioning sequencing situations, where jobs arrive in ordered batches, with privileges given to earlier batches.

Motivated by real-world scenarios where the internal order of jobs within a batch is uncertain (e.g., vehicles arriving on tow trucks in a workshop), this paper develops a new model: sequencing situations with batch-ordered jobs. To address cost allocation under partial order information, two pessimistic cooperative cost games are defined. In the first model, coalition jobs are placed at the end of their respective batches and are not allowed to exchange positions with jobs from other batches. In the second model, after being placed last within their batches, coalition jobs may further rearrange their positions through admissible swaps.

The total cost of the grand coalition, and thus the allocation among agents, depends on whether communication among batches is allowed. The paper proposes and axiomatizes two corresponding allocation rules, each yielding core elements of the respective games. Importantly, the model generalizes classical sequencing games: if each batch contains only one agent, it coincides with sequencing games with an initial order; if there is only one batch, it coincides with sequencing games without an initial order. Hence, our results can be readily adapted to these traditional sequencing situations.

The paper is organized as follows. Section 2 revisits key concepts from cooperative game theory and summarizes relevant literature on sequencing games. Section 3 introduces and analyzes sequencing situations with batch-ordered jobs: Subsection 3.1 focuses on models without inter-batch communication, while Subsection 3.2 considers scenarios where coalitions can rearrange jobs across batches and in Subsection 3.3, we define and characterize a cost allocation rule corresponding to each situation. To illustrate the practical applicability of our approach, Section 4 presents a real-world case study involving job prioritization under partial sequencing information. Section 5 presents two generalizations of our model that connect sequencing games with batch-ordered jobs to classical sequencing situations with or without an initial order. Finally, the Appendix contains detailed proofs of technical results.

2. PRELIMINARIES, NOTATION AND LITERATURE REVIEW

Let N be a finite subset of the natural numbers and $S \in 2^N$, we denote by $s = |S|$ its cardinality and for each $i \in N \backslash S$, we denote $S \cup i$ instead of $S \cup \{i\}$. Given $x \in \mathbb{R}^N$, we denote $x(S) = \sum_{i \in S} x_i$ when $S \neq \emptyset$, and $x_S \in \mathbb{R}^S$ the vector x restricted to S. A *cooperative game* is a pair (N, c) where $c : 2^N \to \mathbb{R}$ satisfies $c(\emptyset) = 0$. The elements of N are called *players*, the subsets of N *coalitions*, and c the *characteristic function*. We denote by C^N the set of cooperative games with N as set of players and write $c \in C^N$ instead of (N, c). To simplify, we write $c(i_1, \ldots, i_s)$ instead of $c(\{i_1, \ldots, i_s\})$. Let $c \in C^N$, its corresponding *cost saving game* is defined for each $S \in 2^N$ by $v_c(S) = \sum_{i \in S} c(i) - c(S)$.

We say that an allocation $x \in \mathbb{R}^N$ is *efficient* for a game $c \in C^N$ if $x(N) = c(N)$. The set of all efficient allocations for game c is the hyperplane $H(c) = \{x \in \mathbb{R}^N : x(N) = c(N)\}$. An allocation $x \in \mathbb{R}^N$ is *stable* if $x(S) \leq c(S)$ for all $S \in 2^N$. The set of all stable allocations in the game c is $\mathcal{C}ore(c) = \{x \in H(c) : x(S) \leq c(S) \text{ for all } S \in 2^N\}$ and is called the *core* of c. A game $c \in C^N$ is *balanced* (cf. Bondareva, 1963) if its core is non-empty. An important class of games with non-empty core is the class of concave games. A game $c \in C^N$ is *concave* if $c(S \cup i) - c(S) \geq c(T \cup i) - c(T)$ for all $S, T \in 2^N$ such that $S \subset T \subset N \backslash \{i\}$. The counterpart of concavity for cost savings games is *convexity*. Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N into $m \geq 2$ non-empty subsets. A game $c \in C^N$ is *decomposable* with respect to $\mathcal{P}$ (cf. Shapley, 1971) if for each $S \in 2^N$, $c(S) = \sum_{r=1}^{m} c(S \cap N_r)$. Shapley (1971) shows that a decomposable game is concave if, and only if, each component is concave.

Let $S \in 2^N$, $\Pi(S)$ is the set of orders of S, that is, bijective functions from S to $\{1, \ldots, s\}$. A generic order of S is denoted by $\sigma_S \in \Pi(S)$ where $\sigma_S(i) = l$ means that player i is in position l in the order σ_S. Given $i \in S$ and $\sigma_S \in \Pi(S)$, let $P(i, \sigma_S) = \{j \in S : \sigma_S(j) < \sigma_S(i)\}$ and $F(i, \sigma_S) = \{j \in S : \sigma_S(j) > \sigma_S(i)\}$ be the set of *predecessors* and *followers* of i with respect to σ_S, respectively. Let $S_1, S_2 \in 2^N$ with $S_1 \cap S_2 = \emptyset$, and let $\sigma_{S_1} \in \Pi(S_1)$, $\sigma_{S_2} \in \Pi(S_2)$, the order $\sigma_{S_1 \cup S_2} = (\sigma_{S_1}, \sigma_{S_2}) \in \Pi(S_1 \cup S_2)$ denotes that agents belonging to S_1 are placed before agents belonging to S_2 with $\sigma_{S_1 \cup S_2}(i) = \sigma_{S_1}(i)$ if $i \in S_1$ and $\sigma_{S_1 \cup S_2}(i) = \sigma_{S_2}(i) + s_1$ if $i \in S_2$.

A *sequencing situation* is a triple (N, p, α) and, possibly, some (information on the) initial order, were $N = \{1, \ldots, n\}$ is a finite set of agents, each one owning one job that has to be processed on a machine. To simplify, we identify agent i's job with i. The *processing times* of the jobs are given by $p = (p_i)_{i \in N}$ with $p_i > 0$ for all $i \in N$. Each agent $i \in N$ has a cost function $c_i : [0, \infty) \to \mathbb{R}$. For every $t \in [0, \infty)$, $c_i(t)$ denotes the cost for job i if his completion time is equal to t. We assume that c_i is linear for all $i \in N$. Then, there exists $\alpha_i, \beta_i \geq 0$ such that $c_i(t) = \alpha_i t + \beta_i$ for all $i \in N$, where β_i is the service cost, which is fixed, and $\alpha_i t$ is the *completion cost*.

For any $\sigma \in \Pi(N)$, $C(S, \sigma)$ is the *aggregate (completion) cost of coalition* S in the order σ, formally defined by[1]

$$C(S, \sigma) = \sum_{i \in S} \alpha_i \Big(p_i + \sum_{j \in P(i, \sigma)} p_j \Big).$$

An order that minimizes the aggregate cost of coalition N is called *optimal order* and it is denoted by $\hat{\sigma}$. An optimal order is obtained by ordering jobs in non-increasing order of their *urgency indices*, defined, for each $i \in N$ by $u_i = \frac{\alpha_i}{p_i}$ (Smith, 1956). We denote by $\Omega(N, p, \alpha)$ the set of optimal orders for the sequencing situation (N, p, α) that satisfy the condition that, when two jobs share the same urgency, the one with shortest processing time goes first. Formally, if there exist $i, j \in N$ such that $u_i = u_j$ and $p_i < p_j$, $\hat{\sigma}(i) < \hat{\sigma}(j)$ for all $\hat{\sigma} \in \Omega(N, p, \alpha)$.

[1]Since β_i is fixed for all $i \in N$, we consider $c_i(t) = \alpha_i t$.

A subsequent problem in a sequencing situation is the distribution of the total cost of the optimal order among the agents. To address it, two different approaches concerning the information on the initial order are considered.

- *A sequencing situation with initial order* (cf. Curiel et al., 1989) is a quadruple (N, p, α, σ_0) where $\sigma_0 \in \Pi(N)$ is the initial order of the jobs. We denote by $\mathcal{S}^0$ the class of all sequencing situations with initial order.
- *A sequencing situation without initial order* (cf. Klijn and Sánchez Rodríguez, 2006; Chun, 2006) is a triple (N, p, α) in which there is no information about an initial order. We denote by $\mathcal{S}$ the class of all sequencing situations without initial order.

Given a sequencing situation with initial order $(N, p, \alpha, \sigma_0) \in \mathcal{S}^0$, we say that $\sigma \in \Pi(N)$ is an *admissible order* for coalition $S \in 2^N$ if $P(i, \sigma) = P(i, \sigma_0)$ for all $i \in N \backslash S$. The set of all admissible orders for coalition S is denoted by $\mathcal{A}(S, \sigma_0) \in \Pi(N)$. A coalition $S \in 2^N$ is called *connected* if for all $i, j \in S$ and $k \in N$, $\sigma(i) < \sigma(k) < \sigma(j)$ implies $k \in S$. We say that a coalition S' is a *component of* S if $S' \subset S$, S' is connected, and for every $i \in S \backslash S'$, $S' \cup i$ is not connected. The components of S form a partition of S which we denote by S/σ_0. Curiel et al. (1989) define the gain of swapping i and j by $g_{ij} = \max\{0, \alpha_j p_i - \alpha_i p_j\}$. The σ_0*-sequencing game* (N, c_{σ_0}) (cf. Curiel et al., 1989), is defined by

$$c_{\sigma_0}(S) = \min_{\sigma \in \mathcal{A}(S, \sigma_0)} C(S, \sigma) \text{ for all } S \in 2^N.$$

Following Curiel et al. (1989), (N, c_{σ_0}) is concave[2].

A *cost allocation rule* on $\mathcal{S}^0$ is a mapping ψ that assigns to each sequencing situation with initial order $(N, p, \alpha, \sigma_0) \in \mathcal{S}^0$ a vector $\psi(N, p, \alpha, \sigma_0) \in \mathbb{R}^N$. Hamers (1995) introduced the *Gain Splitting rules* (GS), which generalize the *Equal Gain Splitting rule* (EGS) originally defined in Curiel et al. (1989). Translating this formulation into cost terms, we define the *Cost Splitting rules with initial order* (CSO) as follows: for every $\lambda \in \Lambda$ and $i \in N$,

$$\mathrm{CSO}_i^\lambda(N, p, \alpha, \sigma_0) = C(i, \sigma_0) - \left(\sum_{k \in P(i, \sigma_0)} (1 - \lambda_{ki}) g_{ki} + \sum_{j \in F(i, \sigma_0)} \lambda_{ij} g_{ij} \right).$$

As shown by Hamers (1995), $\mathrm{CSO}^\lambda \in \mathcal{C}ore(c_{\sigma_0})$ for all $\lambda \in \Lambda$.[3]

Next, we consider sequencing situations without initial order. The *tail game* (cf. Klijn and Sánchez Rodríguez, 2006) associated to $(N, p, \alpha) \in \mathcal{S}$ is defined by

$$c_{tail}(S) = C(S, (\sigma_{N \backslash S}, \hat{\sigma}_S)) \text{ for all } S \in 2^N,$$

where $\sigma_{N \backslash S} \in \Pi(N \backslash S)$ and $\hat{\sigma}_S \in \Omega(S, p_S, \alpha_S)$. In such a game, as there is no information about an initial order, coalitions assume they will be processed at the tail of some "artificial" initial order. Klijn and Sánchez Rodríguez (2006) show that the game (N, c_{tail}) is concave.

A *cost allocation rule* on $\mathcal{S}$ is a map ψ that assigns to each sequencing situation without initial order, $(N, p, \alpha) \in \mathcal{S}$, a vector $\psi(N, p, \alpha) \in \mathbb{R}^N$. Klijn and Sánchez Rodríguez (2006) define the *cost splitting rule according to optimal orders* (CS). Formally,

$$\mathrm{CS}(N, p, \alpha) = \Big(\frac{1}{|\Omega(N, p, \alpha)|} \sum_{\hat{\sigma} \in \Omega(N, p, \alpha)} C(i, \hat{\sigma}) \Big)_{i \in N}.$$

They show that $\mathrm{CS}(N, p, \alpha) \in \mathcal{C}ore(c_{tail})$ and characterize the rule in the class of sequencing situations without initial order in which all jobs have the same processing time.

3. SEQUENCING SITUATIONS WITH BATCH-ORDERED JOBS

Let N be a set of agents and $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N with $m \leq n$. We introduce the notion of *batch order*, which is denoted by $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$. A batch order represents an initial order in the partition in which jobs in N_1 are initially placed before jobs in $N \backslash N_1$, jobs in N_2 are initially placed before jobs in $N \backslash (N_1 \cup N_2)$, and so on. The batch order does not provide information about the order of jobs in the same batch. For each $r \in \{1, \ldots, m\}$, we denote $N^r = \bigcup_{q \leq r} N_q$ and, given $i \in N$, we denote by $r(i)$ the index of the element in $\mathcal{P}$ to which i belongs, that is, $r(i) \in \{1, \ldots, m\}$ with $i \in N_{r(i)}$.

[2] Curiel et al. (1989) show that the cost savings game of c_{σ_0} is convex. Such a game is defined by

$$v_{c_{\sigma_0}}(S) = \sum_{S' \in S/\sigma_0} \sum_{\substack{i,j \in S' \\ i \in P(j, \sigma_0)}} g_{ij} \text{ for all } S \in 2^N.$$

[3] Hamers (1995) prove that $\mathrm{GS}^\lambda \in \mathcal{C}ore(v_{c_{\sigma_0}})$ for all $\lambda \in \Lambda$.

Definition 1. Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N. A *sequencing situation with batch-ordered jobs* is a quadruple $(N, p, \alpha, \sigma_{\mathcal{P}})$ where (N, p, α) is a sequencing situation without initial order and $\sigma_{\mathcal{P}}$ is a batch order. We denote by $\mathcal{SB}$ the class of all sequencing situations with batch-ordered jobs.

We study how to distribute the total costs of a sequencing situation with batch-ordered jobs in the framework of cooperative game theory. We consider two different approaches concerning the communication among agents of different batches and we introduce two cost allocation rules.

3.1 Sequencing situations with batch-ordered jobs without communication

In this section, we assume that there is no communication between two agents of two different batches. Then, two jobs of two different batches cannot change their positions, that is, given $i, j \in N$ such that $r(i) < r(j)$, job j can never be processed before job i. Next, we define the *game with batch-ordered jobs without communication.*

Definition 2. Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N and let $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ with $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$. The associated *game with batch-ordered jobs without communication*, $\bar{c}_{\sigma_{\mathcal{P}}} \in C^N$, is defined by

$$\bar{c}_{\sigma_{\mathcal{P}}}(S) = C(S, \sigma_{\mathcal{P}\cap S}) \text{ for all } S \in 2^N,$$

where $\sigma_{\mathcal{P}\cap S} = (\sigma_{N_1\backslash S}, \hat{\sigma}_{N_1\cap S}, \ldots, \sigma_{N_m\backslash S}, \hat{\sigma}_{N_m\cap S}) \in \Pi(N)$ is an order such that, for all $r \in \{1, \ldots m\}$, $\sigma_{N_r\backslash S} \in \Pi(N_r\backslash S)$ and $\hat{\sigma}_{N_r\cap S} \in \Omega(N_r \cap S, p_{N_r\cap S}, \alpha_{N_r\cap S})$.

The following results give insight into some properties of the game with batch-ordered jobs without communication.

Proposition 3. *Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N and let $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ with $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$,*

1. *$\bar{c}_{\sigma_{\mathcal{P}}}(S) = \sum_{r=1}^{m} c^r_{tail}(N_r \cap S)$ for all $S \in 2^N$, where*

$$c^r_{tail}(T) = \begin{cases} c_{tail}(T) & \text{if } r = 1, \\ \sum_{i\in T}\alpha_i \sum_{j\in N^{r-1}} p_j + c_{tail}(T) & \text{if } r > 1, \end{cases}$$

   for all $T \in 2^{N_r}$, $r \in \{1, \ldots, m\}$.

2. *$\bar{c}_{\sigma_{\mathcal{P}}}$ is decomposable with respect to $\mathcal{P}$.*
3. *$\bar{c}_{\sigma_{\mathcal{P}}}$ is concave.*

Proof. If $m = 1$, we deal with a sequencing situation without initial order and, therefore, the results are immediate (cf. Klijn and Sánchez Rodríguez, 2006). Then, we consider $m > 1$.

1. First, we present an alternative method for expressing the characteristic function of $\bar{c}_{\sigma_{\mathcal{P}}}$ as the sum of the batch costs according to $\sigma_{\mathcal{P}\cap S}$. Let $S \in 2^N$,

$$\begin{aligned}\bar{c}_{\sigma_{\mathcal{P}}}(S) = C(S, \sigma_{\mathcal{P}\cap S}) &= \sum_{i\in S}\alpha_i(p_i + \sum_{j\in P(i,\sigma_{\mathcal{P}\cap S})} p_j) = \sum_{r=1}^{m}\sum_{i\in N_r\cap S}\alpha_i(p_i + \sum_{j\in P(i,\sigma_{\mathcal{P}\cap S})} p_j) \\ &= \sum_{r=1}^{m} C(N_r \cap S, \sigma_{\mathcal{P}\cap S}). \qquad (1)\end{aligned}$$

   Now, we show that each element of the above sum corresponds to the characteristic function of the game c^r_{tail}. For all $r \in \{2, \ldots, m\}$,

$$\begin{aligned} c^r_{tail}(N_r \cap S) &= \sum_{i\in N_r\cap S}\alpha_i \sum_{j\in N^{r-1}} p_j + \sum_{i\in N_r\cap S}\alpha_i \sum_{j\in P(i,\sigma_{\mathcal{P}\cap S})\cap N_r} p_j \\ &= \sum_{i\in N_r\cap S}\alpha_i \sum_{j\in N^{r-1}\cup\left(P(i,\sigma_{\mathcal{P}\cap S})\cap N_r\right)} p_j = \sum_{i\in N_r\cap S}\alpha_i \sum_{j\in P(i,\sigma_{\mathcal{P}\cap S})} p_j \\ &= C(N_r \cap S, \sigma_{\mathcal{P}\cap S}), \qquad (2)\end{aligned}$$

   where the first equality is true by definition of the c_{tail} game restricted to N_r and the third one follows since $N^{r-1} \cup \left(P(i, \sigma_{\mathcal{P}\cap S}) \cap N_r\right) = (N^{r-1} \cup P(i, \sigma_{\mathcal{P}\cap S})) \cap N^r = P(i, \sigma_{\mathcal{P}\cap S})$.

   Therefore, combining Equations (1) and (2), $\bar{c}_{\sigma_{\mathcal{P}}}(S) = \sum_{r=1}^{m} c^r_{tail}(N_r \cap S)$.

2. We need to prove that, for all $S \in 2^N$, $\bar{c}_{\sigma_\mathcal{P}}(S) = \sum_{r=1}^{m} \bar{c}_{\sigma_\mathcal{P}}(N_r \cap S)$. Let $S \in 2^N$,

$$\bar{c}_{\sigma_\mathcal{P}}(S) = \sum_{r=1}^{m} C(N_r \cap S, \sigma_{\mathcal{P}\cap S}) = \sum_{r=1}^{m} C(N_r \cap S, \sigma_{\mathcal{P}\cap(N_r\cap S)}) = \sum_{r=1}^{m} \bar{c}_{\sigma_\mathcal{P}}(N_r \cap S),$$

where the first equality follows by item 1, the second one since $P(i, \sigma_{\mathcal{P}\cap(N_r\cap S)}) = P(i, \sigma_{\mathcal{P}\cap S})$ for all $i \in N_r \cap S$ and all $r \in \{1, \ldots, m\}$, and the last one by definition of the game with batch-ordered jobs without communication.

3. Following Shapley (1971), a decomposable game is concave if, and only if, each component is concave. Then, we have to show that c^r_{tail} is concave for each $r \in \{1, \ldots, m\}$. This follows since c^r_{tail} is the sum of a non-negative additive game and a tail game which are both concave. Therefore, $\bar{c}_{\sigma_\mathcal{P}}$ is concave.

□

3.2 Sequencing situations with batch-ordered jobs with communication

Next, we consider situations where there is communication between batches. We define the $\sigma_\mathcal{P}$-sequencing game to model this situation. In this game, since there is no information about the order of agents within the same batch, each coalition assumes that its members are processed last in their respective batches, as in the previous model: for any $S \in 2^N$, players belonging to S assume that there is an "artificial" initial order $\sigma_{\mathcal{P}\cap S}$. Nevertheless, as there is communication between batches, given two members of the same coalition in two different batches, they can swap positions if they are in a connected coalition in that initial order (they cannot harm any agent belonging to $N\backslash S$). Finally, coalitions pay bear cost of being processed under the adjusted order.

Definition 4. Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N and let $(N, p, \alpha, \sigma_\mathcal{P}) \in \mathcal{SB}$ with $\sigma_\mathcal{P} = (N_1, \ldots, N_m)$. The associated *$\sigma_\mathcal{P}$-sequencing game*, $c_{\sigma_\mathcal{P}} \in C^N$, is defined by

$$c_{\sigma_\mathcal{P}}(S) = \min_{\sigma \in \mathcal{A}(S, \sigma_{\mathcal{P}\cap S})} C(S, \sigma) \text{ for all } S \in 2^N.$$

The $\sigma_\mathcal{P}$-sequencing game is related to the game with batch-ordered jobs without communication: the value of a coalition is the value of the coalition when communication among batches is not allowed minus the possible cost savings from swapping jobs belonging to different batches.

Lemma 5. *Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N and let $(N, p, \alpha, \sigma_\mathcal{P}) \in \mathcal{SB}$ with $\sigma_\mathcal{P} = (N_1, \ldots, N_m)$. The $\sigma_\mathcal{P}$-sequencing game can be rewritten as*

$$c_{\sigma_\mathcal{P}}(S) = \bar{c}_{\sigma_\mathcal{P}}(S) - v(S) \text{ for all } S \in 2^N,$$

where $v(S) = \sum_{S' \in S/\sigma_{\mathcal{P}\cap S}} \sum_{\substack{i,j \in S' \\ i \in P(j, \sigma_{\mathcal{P}\cap S})}} g_{ij}$ *for all* $S \in 2^N$.

Proof. For all $S \in 2^N$,

$$c_{\sigma_\mathcal{P}}(S) = \min_{\sigma \in \mathcal{A}(S, \sigma_{\mathcal{P}\cap S})} C(S, \sigma) = C(S, \sigma_{\mathcal{P}\cap S}) - \sum_{S' \in S/\sigma_{\mathcal{P}\cap S}} \sum_{\substack{i,j \in S' \\ i \in P(j, \sigma_{\mathcal{P}\cap S})}} g_{ij} = \bar{c}_{\sigma_\mathcal{P}}(S) - v(S),$$

where the second equality is true by definition of the set of all admissible orders of $\sigma_{\mathcal{P}\cap S}$ for coalition S. □

Example 6. Consider the sequencing situation with batch-ordered jobs $(N, p, \alpha, \sigma_\mathcal{P}) \in \mathcal{SB}$ such that $N = \{1, 2, 3, 4\}$, $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$, $\sigma_\mathcal{P} = (\{1, 2\}, \{3, 4\})$, $p = (1, 1, 1, 1)$ and $\alpha = (1, 2, 3, 4)$.

First, we compute $\bar{c}_{\sigma_\mathcal{P}}$. For instance, for coalition $S = \{2, 3\}$ we have $\bar{c}_{\sigma_\mathcal{P}}(S) = C(S, (1, 2, 4, 3)) = 16$. The values for one- and two-player coalitions are shown below:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$\bar{c}_{\sigma_\mathcal{P}}(S)$	2	4	12	16	4	14	18	16	20	24

The remaining values can be obtained by additivity (Proposition 3, item 2):

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$\bar{c}_{\sigma_{\mathcal{P}}}(S)$	16	20	26	28	28

Now, we compute $c_{\sigma_{\mathcal{P}}}$. By definition, $c_{\sigma_{\mathcal{P}}}(S) = \bar{c}_{\sigma_{\mathcal{P}}}(S)$ for all coalitions except $\{1,3,4\}$, $\{2,3,4\}$ and N, where members are allowed to swap positions within admissible rearrangements. Specifically:

- $c_{\sigma_{\mathcal{P}}}(1,3,4) = \min\limits_{\sigma \in \mathcal{A}(\{1,3,4\},(2,1,4,3))} C(\{1,3,4\},\sigma) = C(\{1,3,4\},(2,4,3,1)) = 21,$
- $c_{\sigma_{\mathcal{P}}}(2,3,4) = \min\limits_{\sigma \in \mathcal{A}(\{2,3,4\},(1,2,4,3))} C(\{2,3,4\},\sigma) = C(\{2,3,4\},(1,4,3,2)) = 25,$
- $c_{\sigma_{\mathcal{P}}}(N) = \min\limits_{\sigma \in \mathcal{A}(N,(2,1,4,3))} C(N,\sigma) = C(N,(4,3,2,1)) = 20.$

It turns out that $\sigma_{\mathcal{P}}$-sequencing games are concave. Before we show that the game $v \in C^N$ defined in Lemma 5 is convex. To do that, we need to analyse the impact of the entrance of a player into a coalition regarding positional changes among players of the coalition. Given $S \in 2^N$ and $i \in N \backslash S$, we need to study two possible extra cost savings by swapping jobs from different batches when i joins S:

(i) Those jobs that are in batches $N_{r(i)+1}, \ldots, N_m$; this situation is only possible if $r(i) < m$ and $N_{r(i)+1} \subset S$. In such case, $r^+_{S,i}$ is the largest index with $N_{r(i)+1} \cup \ldots \cup N_{r^+_{S,i}} \subset S$ if $r(i) < m$ and $N_{r(i)+1} \subset S$, and $r^+_{S,i} = 1$ otherwise. Formally,

$$r^+_{S,i} = \begin{cases} \max\{r \in \{r(i)+1,\ldots,m\} : \bigcup\limits_{p=r(i)+1}^{r} N_p \subset S\} & \text{if } r(i) < m \text{ and } N_{r(i)+1} \subset S, \\ 1 & \text{otherwise.} \end{cases}$$

If $r^+_{S,i} > 1$, we denote $M^+_{S,i} = \bigcup\limits_{q=r(i)+1}^{r^+_{S,i}} N_q$.

(ii) Those jobs that are in batches $N_1, \ldots, N_{r(i)-1}$; this situation is only possible if $r(i) > 1$ and $N_{r(i)} \subset S \cup i$. In such case, $r^-_{S,i}$ is the smallest index with $N_{r^-_{S,i}+1} \cup \ldots \cup N_{r(i)} \subset S \cup i$ and $N_{r^-_{S,i}} \cap S \neq \emptyset$ if $r(i) > 1$, and $r^-_{S,i} = m$ otherwise. Formally,

$$r^-_{S,i} = \begin{cases} \min\{r \in \{1,\ldots,r(i)-1\} : \bigcup\limits_{p=r+1}^{r(i)} N_p \subset S \cup i,\ N_r \cap S \neq \emptyset\} & \text{if } r(i) > 1 \text{ and } N_{r(i)} \subset S \cup i, \\ m & \text{otherwise.} \end{cases}$$

If $r^-_{S,i} < m$, we denote $M^-_{S,i} = \bigcup\limits_{q=r^-_{S,i}}^{r(i)-1} N_q$.

In Figure 1, we show an example of a sequencing situation with batch-ordered jobs in which $r^+_{S,i} > 1$ and $r^-_{S,i} < m$.

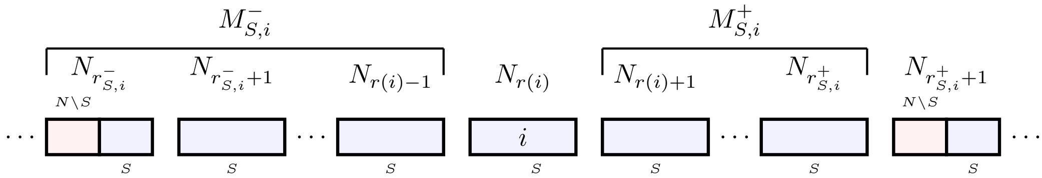

Figure 1: Sequencing situation with batch-ordered jobs in which $r^+_{S,i} > 1$ and $r^-_{S,i} < m$.

Definition 7. Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N, $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ with $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$. Let $S \in 2^N$ and $i \in N \backslash S$. We say that i is a connector in S if there exists $j \in N_{r(i)-1} \cap S$, $k \in N_{r(i)} \cap S$ and $T \in (S \cup i)/\sigma_{\mathcal{P} \cap S}$ such that $i, j, k \in T$.

We now examine the marginal contribution depending on whether i acts as a connector in S or not.

- If i is not connector in S, when i enters coalition S, additional cost savings may arise by neighbour switches of i with agents in $M^+_{S,i}$. Then,

$$v(S \cup i) - v(S) = \begin{cases} 0 & \text{if } r^+_{S,i} = 1, \\ \sum\limits_{k \in M^+_{S,i}} g_{ik} & \text{otherwise.} \end{cases} \tag{3}$$

In Figure 2 we illustrate graphically a situation in which $r^+_{S,i} > 1$.

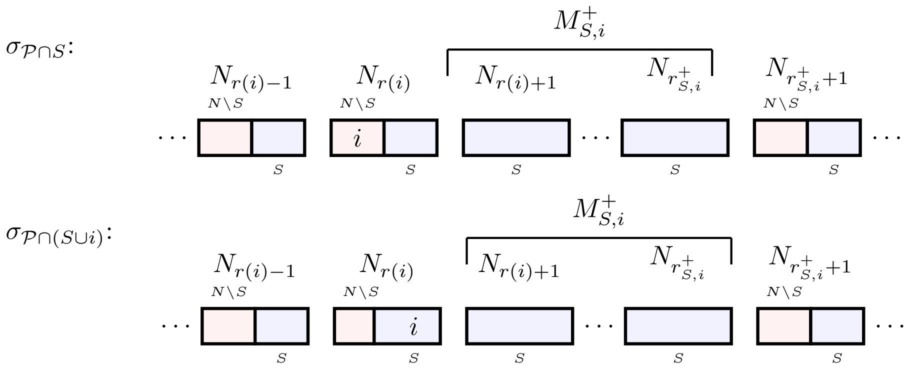

Figure 2: Orders $\sigma_{\mathcal{P} \cap S}$ and $\sigma_{\mathcal{P} \cap (S \cup i)}$ when i is not connector in S.

- If i is connector in S, when i enters coalition S, new cost savings may be possible by neighbour switches of i with agents in $M^-_{S,i} \cup M^+_{S,i}$ and by neighbour switches of agents in $M^-_{S,i}$ with agents in $M^+_{S,i}$. Then,

$$v(S \cup i) - v(S) = \begin{cases} \sum\limits_{j \in S \cap M^-_{S,i}} \sum\limits_{k \in N_{r(i)}} g_{jk} & \text{if } r^+_{S,i} = 1, \\ \sum\limits_{j \in S \cap M^-_{S,i}} \sum\limits_{k \in M^+_{S,i} \cup N_{r(i)}} g_{jk} + \sum\limits_{k \in M^+_{S,i}} g_{ik} & \text{otherwise.} \end{cases} \tag{4}$$

In Figure 3 we illustrate graphically a situation in which $r^+_{S,i} > 1$.

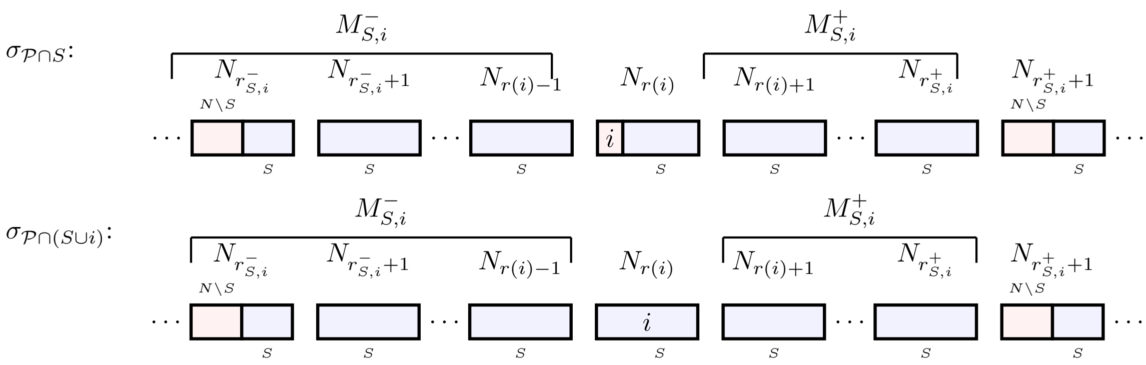

Figure 3: Orders $\sigma_{\mathcal{P} \cap S}$ and $\sigma_{\mathcal{P} \cap (S \cup i)}$ when i is connector in S.

Proposition 8. *Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N and let $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ with $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$. The game $v \in C^N$, defined by $v(S) = \sum\limits_{S' \in S/\sigma_{\mathcal{P} \cap S}} \Big(\sum\limits_{\substack{i,j \in S' \\ i \in P(j, \sigma_{\mathcal{P} \cap S})}} g_{ij} \Big)$ for all $S \in 2^N$, is convex.*

Proof. Let $i \in N$ and $S \subset T \subset N \setminus \{i\}$. We must show $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$.

From the definitions we note the following: (1) If i is a connector in S, then i is also a connector in T (but not necessarily the reverse). (2) The sets of agents with whom i (or its neighbors) can swap increase with the coalition: $M^+_{S,i} \subseteq M^+_{T,i}$ and $M^-_{S,i} \subseteq M^-_{T,i}$. (3) All swap gains g_{ab} are nonnegative.

Considering separately the possible cases for i (non-connector vs. connector, and the value of $r^+_{S,i}$), it follows that $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$. Therefore, v is convex. □

The following result is a direct consequence of Lemma 5 and Propositions 3 and 8. From Lemma 5, we obtain that $c_{\sigma_{\mathcal{P}}} = \bar{c}_{\sigma_{\mathcal{P}}} - v$ and concavity of $\bar{c}_{\sigma_{\mathcal{P}}}$ (Proposition 3) and convexity of v (Proposition 8) imply concavity of $c_{\sigma_{\mathcal{P}}}$. The proof is, therefore, omitted.

Theorem 9. *Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N and let $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ with $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$. The game $(N, c_{\sigma_{\mathcal{P}}})$ is concave.*

3.3 Cost allocation rules

Having described the main properties of the games, we now turn to the problem of finding (intuitive) cost allocation rules for the class of sequencing situations with batch-ordered jobs. A *cost allocation rule* on the class of sequencing situations with batch-ordered jobs is a map ψ that assigns to each $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ a vector $\psi(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathbb{R}^N$. We denote by $\Omega(N, p, \alpha, \sigma_{\mathcal{P}})$ the set of optimal orders of the problem where the initial order of the batches is respected, that is, if $\hat{\sigma}_{\mathcal{P}} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$, then $\hat{\sigma}_{\mathcal{P}} = (\hat{\sigma}_{N_1}, \ldots, \hat{\sigma}_{N_m})$ being $\hat{\sigma}_{N_k} \in \Omega(N_k, p_{N_k}, \alpha_{N_k})$ for all $k \leq m$. Next, we define two rules.

Definition 10. We define the *batch-ordered jobs without communication rule*, BNC, and the *batch-ordered jobs with communication rule*, BC, for each partition of N, $\mathcal{P} = \{N_1, \ldots, N_m\}$, and each $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ with $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$, by

$$\begin{aligned}
\mathrm{BNC}(N, p, \alpha, \sigma_{\mathcal{P}}) =& \Big(\frac{1}{|\Omega(N, p, \alpha, \sigma_{\mathcal{P}})|} \Big[\sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} C(i, \hat{\sigma}_{\mathcal{P}}) \Big] \Big)_{i \in N}, \text{ and} \\
\mathrm{BC}(N, p, \alpha, \sigma_{\mathcal{P}}) =& \Big(\frac{1}{|\Omega(N, p, \alpha, \sigma_{\mathcal{P}})|} \Big[\sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} \Big(C(i, \hat{\sigma}_{\mathcal{P}}) - \sum_{l \in P(i, \hat{\sigma}_{\mathcal{P}})} g_{li} \Big) \Big] \Big)_{i \in N}.
\end{aligned}$$

When employing the BNC rule, each agent pays the average of its aggregate cost in the set of optimal orders in which the initial order of the batches is respected. When employing the BC rule, for each order in $\Omega(N, p, \alpha, \sigma_{\mathcal{P}})$, after such distribution, each agent evaluates whether there exists another agent initially positioned ahead of it with a lower urgency. If such an agent exists, the savings resulting from reordering are subtracted from the payment of the agent who initially occupied the worst position. Finally, each agent pays the average of its aggregate cost.

Next, we show that the rules defined above are core allocations of the games previously defined. Before showing that, we relate the cost of a coalition $S \in 2^N$ in the orders $\hat{\sigma}_{\mathcal{P}}$ and $\sigma_{\mathcal{P} \cap S}$.

Lemma 11. *Let $\mathcal{P} = \{N_1, \ldots, N_m\}$ be a partition of N and let $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ with $\sigma_{\mathcal{P}} = (N_1, \ldots, N_m)$, $C(S, \hat{\sigma}_{\mathcal{P}}) \leq C(S, \sigma_{\mathcal{P} \cap S})$ for all $S \in 2^N$ and $\hat{\sigma}_{\mathcal{P}} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$.*

Proof. Let $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$, $S \in 2^N$ and $\hat{\sigma}_{\mathcal{P}} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$. First, for all $r \in \{1, \ldots, m\}$,

$$\begin{aligned}
C(N_r \cap S, \hat{\sigma}_{\mathcal{P}}) =& \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in P(i, \hat{\sigma}_{\mathcal{P}})} p_j = \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in N^{r-1}} p_j + \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in P(i, \hat{\sigma}_{\mathcal{P}}) \cap N_r} p_j \\
=& \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in N^{r-1}} p_j + \sum_{i \in N_r \cap S} \alpha_i \sum_{\substack{j \in N_r \\ u_j > u_i}} p_j \\
=& \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in N^{r-1}} p_j + \sum_{i \in N_r \cap S} \alpha_i \sum_{\substack{j \in N_r \cap (N \setminus S) \\ u_j > u_i}} p_j + \sum_{i \in N_r \cap S} \alpha_i \sum_{\substack{j \in N_r \cap S \\ u_j > u_i}} p_j \\
\leq& \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in N^{r-1}} p_j + \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in N_r \cap (N \setminus S)} p_j + \sum_{i \in N_r \cap S} \alpha_i \sum_{\substack{j \in N_r \cap S \\ u_j > u_i}} p_j \\
=& \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in N^{r-1}} p_j + \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in P(i, \sigma_{\mathcal{P} \cap S}) \cap N_r} p_j = \sum_{i \in N_r \cap S} \alpha_i \sum_{j \in P(i, \sigma_{\mathcal{P} \cap S})} p_j \\
=& C(N_r \cap S, \sigma_{\mathcal{P} \cap S}),
\end{aligned}$$

where the fifth equality is true by definition of $\sigma_{\mathcal{P} \cap S}$. Then, applying the last equation,

$$C(S, \hat{\sigma}_{\mathcal{P}}) = \sum_{r=1}^{m} C(N_r \cap S, \hat{\sigma}_{\mathcal{P}}) \leq \sum_{r=1}^{m} C(N_r \cap S, \sigma_{\mathcal{P} \cap S}) = C(S, \sigma_{\mathcal{P} \cap S}).$$

□

Theorem 12. *BNC*$(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{C}ore(\bar{c}_{\sigma_{\mathcal{P}}})$ *for all* $(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{SB}$ *and BC*$(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{C}ore(c_{\sigma_{\mathcal{P}}})$ *for all* $(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{SB}$.

Proof. Let $(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{SB}$. Efficiency follows by efficiency without communication of BNC and by efficiency of BC.

Next, we prove stability for BNC. Let $x = \text{BNC}(N,p,\alpha,\sigma_P)$, we show that $x(S) \leq \bar{c}_{\sigma_{\mathcal{P}}}(S)$ for all $S \in 2^N$. Let $S \in 2^N$,

$$\begin{aligned} x(S) =& \frac{1}{|\Omega(N,p,\alpha,\sigma_{\mathcal{P}})|} \sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} C(S,\hat{\sigma}_{\mathcal{P}}) \\ \leq& \frac{1}{|\Omega(N,p,\alpha,\sigma_{\mathcal{P}})|} \sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} C(S,\sigma_{\mathcal{P}\cap S}) = C(S,\sigma_{\mathcal{P}\cap S}) = \bar{c}_{\sigma_{\mathcal{P}}}(S), \end{aligned}$$

where the inequality is true by Lemma 11.

It remains to prove stability for BC. Let $x = \text{BC}(N,p,\alpha,\sigma_P)$ and $S \in 2^N$,

$$x(S) = \frac{1}{|\Omega(N,p,\alpha,\sigma_{\mathcal{P}})|} \sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} \Big(C(S,\hat{\sigma}_{\mathcal{P}}) - \sum_{i\in S} \sum_{l \in P(i,\hat{\sigma}_{\mathcal{P}})} g_{li} \Big).$$

We show that $x(S) \leq c_{\sigma_{\mathcal{P}}}(S)$ for all $S \in 2^N$. Before that, by Lemma 11, $c(S,\hat{\sigma}_{\mathcal{P}}) \leq c(S,\sigma_{\mathcal{P}\cap S})$ for all $\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})$ and,

$$\sum_{i\in S} \sum_{l\in P(i,\hat{\sigma}_{\mathcal{P}})} g_{li} \geq \sum_{\substack{l,i\in S \\ l\in P(i,\hat{\sigma}_{\mathcal{P}})}} g_{li} = \sum_{\substack{l,i\in S \\ l\in P(i,\sigma_{\mathcal{P}\cap S})}} g_{li} \geq \sum_{S'\in S/\sigma_{\mathcal{P}\cap S}} \sum_{\substack{l,i\in S' \\ l\in P(i,\sigma_{\mathcal{P}\cap S})}} g_{li}, \tag{5}$$

where the equality is true since $P(i,\hat{\sigma}_{\mathcal{P}}) \cap S = P(i,\sigma_{\mathcal{P}\cap S}) \cap S$. Therefore, using Lemma 11 and Equation (5),

$$\begin{aligned} x(S) =& \frac{1}{|\Omega(N,p,\alpha,\sigma_{\mathcal{P}})|} \sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} \Big(C(S,\hat{\sigma}_{\mathcal{P}}) - \sum_{i\in S} \sum_{l\in P(i,\hat{\sigma}_{\mathcal{P}})} g_{li} \Big) \\ \leq& \frac{1}{|\Omega(N,p,\alpha,\sigma_{\mathcal{P}})|} \sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} \Big(C(S,\sigma_{\mathcal{P}\cap S}) - \sum_{i\in S} \sum_{l\in P(i,\hat{\sigma}_{\mathcal{P}})} g_{li} \Big) \\ \leq& \frac{1}{|\Omega(N,p,\alpha,\sigma_{\mathcal{P}})|} \sum_{\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})} \Big(C(S,\sigma_{\mathcal{P}\cap S}) - \sum_{S'\in S/\sigma_{\mathcal{P}\cap S}} \sum_{\substack{l,i\in S' \\ l\in P(i,\sigma_{\mathcal{P}\cap S})}} g_{li} \Big) \\ =& C(S,\sigma_{\mathcal{P}\cap S}) - \sum_{S'\in S/\sigma_{\mathcal{P}\cap S}} \sum_{\substack{l,i\in S' \\ l\in P(i,\sigma_{\mathcal{P}\cap S})}} g_{li} = c_{\sigma_{\mathcal{P}}}(S). \end{aligned}$$

□

Next, we characterize the rules. To do that, we introduce four properties for a cost allocation rule on $\mathcal{SB}$. Given a sequencing situation with batch-ordered jobs $(N,p,\alpha,\sigma_{\mathcal{P}})$, a cost allocation rule for a sequencing situation with batch-ordered jobs ψ satisfies

- *efficiency* if for all $(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{SB}$, $\sum_{i\in N} \psi_i(N,p,\alpha,\sigma_{\mathcal{P}}) = C(N,\hat{\sigma})$ for any $\hat{\sigma} \in \Omega(N,p,\alpha)$;
- *efficiency without communication* if for all $(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{SB}$, $\sum_{i\in N} \psi_i(N,p,\alpha,\sigma_{\mathcal{P}}) = C(N,\hat{\sigma}_{\mathcal{P}})$ for any $\hat{\sigma}_{\mathcal{P}} \in \Omega(N,p,\alpha,\sigma_{\mathcal{P}})$;
- *equal treatment of equals in batches* if for all $(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{SB}$, $i,j \in N$, $i \neq j$, with $r(i) = r(j)$, $\alpha_j = \alpha_i$, and $p_j = p_i$, then $\psi_i(N,p,\alpha,\sigma_{\mathcal{P}}) = \psi_j(N,p,\alpha,\sigma_{\mathcal{P}})$;
- *urgency by batches* if for all $(N,p,\alpha,\sigma_{\mathcal{P}}) \in \mathcal{SB}$, $i,j \in N$ satisfying one of the following conditions:
  - (i) $r(j) > r(i)$,
  - (ii) $r(j) = r(i)$ and $u_j < u_i$,
  - (iii) $r(j) = r(i)$, $u_j = u_i$ and $p_j > p_i$,

  $\psi_i(N\backslash\{j\}, p_{N\backslash\{j\}}, \alpha_{N\backslash\{j\}}, \sigma_{\mathcal{P}}) = \psi_i(N,p,\alpha,\sigma_{\mathcal{P}})$.

Theorem 13. *The BNC rule is the unique rule on $\mathcal{SB}$ that satisfies the properties of efficiency without communication, equal treatment of equals in batches and urgency by batches.*

Proof. Let $(N, p, \alpha, \sigma_{\mathcal{P}})$ be a sequencing situation with batch-ordered jobs. It is easily seen that BNC satisfies efficiency without communication.

Now, we show that the BNC rule satisfies equal treatment of equals in batches. Let $i, j \in N$, $i \neq j$, such that $\alpha_i = \alpha_j$, $p_i = p_j$ and $r(i) = r(j)$. For each $\hat{\sigma} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$, there exists $\hat{\sigma}' \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$ such that $\hat{\sigma}'(i) = \hat{\sigma}(j)$, $\hat{\sigma}'(j) = \hat{\sigma}(i)$ and $\hat{\sigma}'(k) = \hat{\sigma}(k)$ for each $k \in N \backslash \{i, j\}$. Therefore, $C(i, \hat{\sigma}) = C(j, \hat{\sigma}')$ and it follows that $\text{BNC}_i(N, p, \alpha, \sigma_{\mathcal{P}}) = \text{BNC}_j(N, p, \alpha, \sigma_{\mathcal{P}})$. Applying the definition of BNC, it follows that equal treatment of equals in batches holds.

Next, we show that it satisfies urgency by batches. Let $r \leq m$ and $i, j \in N$. If $r(j) > r(i)$ it follows directly by definition of $\hat{\sigma}_{\mathcal{P}}$. If $r(j) = r(i)$ and $u_j < u_i$ or $r(j) = r(i)$, $u_j = u_i$ and $p_j > p_i$, for each $\hat{\sigma} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$, we consider the order $\hat{\sigma}_{N \backslash \{j\}}$ defined by $\hat{\sigma}_{N \backslash \{j\}}(k) = \hat{\sigma}(k)$ if $k \in P(j, \hat{\sigma})$ and $\hat{\sigma}_{N \backslash \{j\}}(k) = \hat{\sigma}(k) - 1$ if $k \in (N \backslash \{j\}) \backslash P(j, \hat{\sigma})$. By definition of $\Omega(N, p, \alpha, \sigma_{\mathcal{P}})$, for each $\hat{\sigma} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$, $C(i, \hat{\sigma}_{N \backslash \{j\}}) = C(i, \hat{\sigma})$ since $\hat{\sigma}(i) < \hat{\sigma}(j)$. Then, $\Omega(N \backslash \{j\}, p_{N \backslash \{j\}}, \alpha_{N \backslash \{j\}}, \sigma_{\mathcal{P}}) = \{\hat{\sigma}_{N \backslash \{j\}} : \hat{\sigma} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})\}$ and $\text{BNC}_i(N \backslash \{j\}, p_{N \backslash \{j\}}, \alpha_{N \backslash \{j\}}, \sigma_{\mathcal{P}}) = \text{BNC}_i(N, p, \alpha, \sigma_{\mathcal{P}})$. Applying the definition of BNC, it follows that urgency by batches holds.

Finally, we show uniqueness. Let ψ be a rule satisfying efficiency without communication, equal treatment of equals in batches and urgency by batches. For each $r \in \{1, \ldots, m\}$, let $\mathcal{S}_r = \{S_{r,1}, \ldots, S_{r,q}\}$ be the partition of N_r satisfying

1. $u_i = u_j$, $p_i = p_j$ for each $i, j \in S_{r,w}$, $w \in \{1, \ldots, q\}$.
2. $u_i \geq u_j$ for each $i \in S_{r,w}$, $j \in S_{r,w+1}$, $w \in \{1, \ldots, q-1\}$.
3. $p_i < p_j$ for each $i \in S_{r,w}$, $j \in S_{r,w+1}$ with $u_i = u_j$, $w \in \{1, \ldots, q-1\}$.

Then, $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_m\}$ is a partition of N. We show that $\psi(N, p, \alpha, \sigma_{\mathcal{P}}) = \text{BNC}(N, p, \alpha, \sigma_{\mathcal{P}})$ by induction on $s = |\mathcal{S}| = |\mathcal{S}_1| + \ldots + |\mathcal{S}_m|$.

If $s = 1$, it follows that $\mathcal{P} = \{N\}$, $\alpha_i = \alpha_j$ and $p_i = p_j$ for all $i, j \in N$. By efficiency without communication,

$$\sum_{i \in N} \psi_i(N, p, \alpha, \sigma_P) = C(N, \hat{\sigma}_{\mathcal{P}}),$$

where $\hat{\sigma}_{\mathcal{P}} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$. By equal treatment of equals in batches $\psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) = \psi_j(N, p, \alpha, \sigma_{\mathcal{P}})$ for each $i, j \in N$. Combined with efficiency without communication, it follows

$$\psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) = \frac{C(N, \hat{\sigma}_{\mathcal{P}})}{n} = \text{BNC}_i(N, p, \alpha, \sigma_{\mathcal{P}}) \text{for all} i \in N.$$

Hence, the result holds for $s = 1$.

Next, we proceed with the induction step. Let $l \in \{2, \ldots, n\}$, assume $\psi(N, p, \alpha, \sigma_P) = \text{BNC}(N, p, \alpha, \sigma_{\mathcal{P}})$ if $s \leq l - 1$ and let $s = l$. Let $\hat{\sigma}_{\mathcal{P}} \in \Omega(N, p, \alpha, \sigma_{\mathcal{P}})$. By the property of urgency by batches,

$$\psi_{N \backslash S_l}(N, p, \alpha, \sigma_{\mathcal{P}}) = \psi(N \backslash S_l, p_{N \backslash S_l}, \alpha_{S \backslash S_l}, \sigma_{\mathcal{P}}).$$

By the induction hypothesis, it follows

$$\psi_{N \backslash S_l}(N, p, \alpha, \sigma_{\mathcal{P}}) = \psi(N \backslash S_l, p_{N \backslash S_l}, \alpha_{N \backslash S_l}, \sigma_{\mathcal{P}}) = \text{BNC}(N \backslash S_l, p_{N \backslash S_l}, \alpha_{N \backslash S_l}, \sigma_{\mathcal{P}}).$$

It remains to prove that $\psi_{S_l}(N, p, \alpha, \sigma_{\mathcal{P}}) = \text{BNC}_{S_l}(N, p, \alpha, \sigma_{\mathcal{P}})$. By efficiency without communication,

$$\sum_{i \in N \backslash S_l} \psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) = \sum_{i \in N \backslash S_l} \psi_i(N \backslash S_l, p_{N \backslash S_l}, \alpha_{S \backslash S_l}, \sigma_{\mathcal{P}}) = C(N \backslash S_l, \hat{\sigma}_{\mathcal{P}}), \tag{6}$$

and

$$\sum_{i \in N} \psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) = C(N, \hat{\sigma}_{\mathcal{P}}) = C(N \backslash S_l, \hat{\sigma}_{\mathcal{P}}) + C(S_l, \hat{\sigma}_{\mathcal{P}}). \tag{7}$$

Then, using (6) and (7),

$$\sum_{i \in S_l} \psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) = \sum_{i \in N} \psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) - \sum_{i \in N \backslash S_l} \psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) = C(S_l, \hat{\sigma}_{\mathcal{P}}).$$

To conclude, by equal treatment of equals in batches,

$$\psi_i(N, p, \alpha, \sigma_{\mathcal{P}}) = \frac{C(S_l, \hat{\sigma}_{\mathcal{P}})}{|S_l|} = \text{BNC}_i(N, p, \alpha, \sigma_{\mathcal{P}}) \text{ for all } i \in S_l.$$

By mathematical induction, it follows $\psi = \text{BNC}$.

$\square$

The characterization of the BC rule follows from arguments analogous to Theorem 13, so the proof is omitted.

Theorem 14. *The BC rule is the unique rule on $\mathcal{SB}$ that satisfies the properties of efficiency, equal treatment of equals in batches and urgency by batches.*

4. EXAMPLE: A POWER OUTAGE IN VIGO

This section illustrates the practical implications of the sequencing models through a real-world scenario involving a power outage in the city of Vigo, Spain. Suppose that there are six affected factories, which are located across three neighborhoods: Valladares, Teis, and Vigo Centre. A remote repair company is contracted to restore electricity. Each repair takes one hour per factory, and each factory suffers a different hourly loss due to the interruption. The longer a factory remains without power, the higher its total cost in terms of lost production. Each factory has a different hourly loss: $\alpha = (60, 50, 40, 30, 20, 10)$. The total cost resulting from the chosen repair sequence must be shared among the factories. Consequently, the repair company is dealing with a sequencing situation with six jobs. The question is: how should this cost be allocated fairly, depending on what we know about the order of job arrivals? We analyze three scenarios:

1. The power outage occurs simultaneously across all neighborhoods, and there is no predetermined priority among the factories. This corresponds to a standard sequencing situation without initial order, defined as $(N, p, \alpha) \in S$, where $N = \{1, 2, 3, 4, 5, 6\}, p = (1, 1, 1, 1, 1, 1)$, and $\alpha = (60, 50, 40, 30, 20, 10)$. The optimal order is $\hat{\sigma} = (1, 2, 3, 4, 5, 6)$, and the cost allocation can be determined via the CS rule.
2. An initial order among neighborhoods is established: factories in Teis are affected first, followed by those in Valladares, and then those in Vigo Centre. Consequently, there exists an initial order among the neighbourhoods, and we are now dealing with a sequencing situation with batch-ordered jobs $(N, p, \alpha, \sigma_{\mathcal{P}}) \in \mathcal{SB}$ where (N, p, α) are the same as above and $\sigma_{\mathcal{P}} = (\{3, 5\}, \{1\}, \{2, 4, 6\})$. Within this scenario, two cases are considered: one where factories cannot negotiate reordering across batches (using the BNC rule) and another where inter-batch communication is allowed (using the BC rule).
3. The order within each neighborhood is predetermined: in Teis, Factory 5 is affected before Factory 3, and in Vigo Centre the order is Factory 4, then Factory 6, and finally, Factory 2. Hence, this constitutes a sequencing situation with initial order $(N, p, \alpha, \sigma_0) \in \mathcal{S}^0$ where (N, p, α) are the same as above and $\sigma_0 = (5, 3, 1, 4, 6, 2)$. The cost could be distributed using the CSO$^\lambda$ rule with $\lambda_{ij} = 0$ for all $i, j \in N$.

Table 1 summarizes the cost allocation for each factory under the different sequencing rules. Under the CS, the BC and the CSO^0 rules, the total cost is 560, while the BNC rule results in a higher total cost of 670 due to the rigidity of the imposed batch order. This example shows that the cost borne by each factory depends on its position in the initial order considered. In situations without an initial order, we consider the optimal order; in situations with batch-ordered jobs, we consider the optimal order where the initial order of the batches is respected; and in situations with an established initial order, we adhere to that predetermined order.

Rule	Factory 1	Factory 2	Factory 3	Factory 4	Factory 5	Factory 6	**Total Cost**
CS	60	100	120	120	100	60	560
BNC	180	200	40	150	40	60	670
BC	120	160	40	140	40	60	560
CSO$_\lambda$	120	200	60	110	20	50	560

Table 1: Cost allocation for each factory under different sequencing scenarios.

5. CONNECTIONS WITH CLASSICAL SEQUENCING MODELS

In this work, we have analyzed sequencing situations involving batch-ordered jobs from a game-theoretical perspective. Our study has provided a comprehensive framework for understanding how costs can be allocated in such scenarios while ensuring stability and efficiency. In this section, we show how our results can be seen as a generalization of two classic situations: sequencing situations with an initial order and those without an initial order.

First, when each batch consists of a single job, it coincides with sequencing situations with an initial order. In this case, the BC rule reduces to CSO$^\lambda$ with $\lambda_{ij} = 0$ for all $i, j \in N$, uniquely characterized by the properties of efficiency and urgency. Second, when all jobs form a single batch, the model coincides with sequencing situations without an initial order. Here, BC = BNC = CS, which is uniquely characterized by efficiency, equal treatment of equals, and urgency.

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